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A Quantum Formulation of the Relaxation Process and Transport Coefficients of Magnetized Plasma

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From a quantum collective approach, the momentum relaxation time, through both electron–electron and electron–ion interactions, is obtained based on electron wave functions interacting with the continuum oscillations (plasma waves). The theoretical model presented gives a consistent and complete set of transport coefficients for a dense magnetized plasma. This unified scheme of long- and short-range interactions gives conductivity formulas which are free from the usual Debye length, which loses its physical meaning as an upper impact parameter for relatively high-density, coupled plasma.

1. INTRODUCTION

We present a theoretical analysis on transport coefficients of coupled, fully ionized, classical and magnetized plasmas.

Transport phenomena, where understanding has been attempted via the model of discrete interacting particles, i.e., where electrons are elastically deflected at the surface of the Fermi sphere by the ionic density fluctuations, do not give rigorous account of an essential characteristic property of such a medium: excitation of collective oscillations. For that, collective behavior effects are, simply, treated through Coulomb collisions and separated as long- and short-range interactions and arbitrary split in the integration process.

A number of investigators (Marshak, 1941; Hubbard, 1966) developed transport theories for stellar interiors, using an ordinary two-body Boltzmann equation for electron–ion scattering with Born approximation for the unshielded Coulomb potential. In order to eliminate the long-range

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Coulomb potential divergence, the Coulomb potential has been cut off at the mean interionic distance. These models assumed static shielding.

Moreover, the Boltzmann equation commonly used in connection with transport theories is expressed in terms of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$. The point of view is either classical or semiclassical since it assumed that the position ($\mathbf{r}$) and the velocity ($\mathbf{v}$) (momentum of the particles) can be defined simultaneously. If quantum effects are important in the many-particle system, which is the case in dense plasmas, it is necessary to introduce for the relaxation process another variable as suggested first by Wigner (Rozsnyai, 1972). The variable could either be the energy E or equivalently $\hbar\omega$.

2. RELAXATION TIME FROM COLLECTIVE MODES

The plasma under consideration is a continuum of volume Ω containing N electrons and N/Z ions, and the electron density is $n = N/\Omega$. The system exhibits $3N$ (high-frequency branch) and $3N/Z$ (low-frequency branch) characteristic frequencies $\omega_s(\mathbf{q})$ of longitudinal oscillations ($s = e, i$).

The high-frequency branch corresponds to the electron plasma oscillations and the low-frequency branch to the ion sound waves.

The motion of electrons in a coupled plasma is affected by the continuum oscillations (many-body interactions). When such oscillations are excited, each individual particle suffers a small perturbation of its velocity and position, arising from the combined potential of all the other particles.

The plasma oscillations are quasiparticles, plasmons, and ion sound waves which obey Bose–Einstein statistics and their distribution function is

$$\bar{N}_q = \{\exp[\hbar\omega_s(\mathbf{q})/k_B T] - 1\}^{-1}, \qquad s = e, i \tag{1}$$

The electron (as a wave) interacting with the whole plasma can emit and absorb the quasiparticles with energy $\hbar\omega_s(\mathbf{q})$ and momentum $\hbar\mathbf{q}$ (Pines, 1956), such that the perturbed distribution function is relaxed in the process. The oscillation frequencies are assumed to obey the dispersion relations (Klimontovich and Silin, 1960) of a classical plasma.

The kinetic equation for the distribution function f of electrons will satisfy the equality

$$\left.\frac{\partial f}{\partial t}\right|_{\text{field}} = -\left.\frac{\partial f}{\partial t}\right|_{\text{coll}} \tag{2}$$

where the RHS is the collision term for both electron–electron (e–e) and electron–ion ($e-i$) interactions and defines a relaxation time τ_c such that

$$\left.\frac{\partial f}{\partial t}\right|_{\text{coll}} = \sum_{s=e,i} \left.\frac{\partial f}{\partial t}\right|_{e,s} = -\frac{f_1}{\tau_c} \tag{3}$$

In a first-order perturbation f_1 can be put in the form

$$f_1 = \Phi\, \partial f_0/\partial E \tag{4}$$

f_0 is the Fermi–Dirac distribution function and Φ is an arbitrary trial function of the energy E of the electron which will be defined shortly.

The interaction integral is taken to be the collision term of the Bloch transport equation (Haug, 1972)

$$\begin{aligned}\left.\frac{\partial f}{\partial t}\right|_{e,s} = -\frac{\Phi}{8\pi^2 m_s n_s k_B T} \int q^2 \frac{|U_s(\mathbf{q})|^2}{\omega_s(\mathbf{q})} \bar{N}_q \{ & f_0(\mathbf{k}')[1-f_0(\mathbf{k})] \\ & \times \delta(E'-E+\hbar\omega_s) + f_0(\mathbf{k})[1-f_0(\mathbf{k}')]\delta(E'-E-\hbar\omega_s)\} \\ & \times \left[1 - \frac{\Phi(\mathbf{k}')}{\Phi(\mathbf{k})}\right] d^3\mathbf{k}'\end{aligned} \tag{5}$$

$|U_s(\mathbf{q})|^2$ is the square of the Fourier transform of $U_s(\mathbf{r})$, which is a shielded Yukawa potential.

Moreover, the electron energy E is related to the wave vector $\mathbf{k}$ by $E = \hbar^2 k^2/2m$, assuming the Fermi surfaces are spherical.

The relaxation time τ_c for both interactions is taken to be $\tau_c^{-1} = \sum_s \tau_{es}^{-1}$ ($s = e, i$) and used in the usual definition of the current density and heat flux density to yield the transport coefficients.

From equations (2) and (4) we have

$$\tau_{es}(E) = \Phi(\partial f_0/\partial E)/(\partial f/\partial t)|_{es} \tag{6}$$

The unknown function $\Phi = e|\mathbf{E}|v_x C(E)$ can be evaluated by assuming that the fields are in the x direction, i.e., $\mathbf{E} = (\xi, 0, 0)$ and v_x is the electron speed component which is parallel to $\mathbf{E}$.

Since $C(E)$ is still an unknown function to be defined, for that Φ will be used as a trial function through a Kohler variational principle (Haug, 1972) thereby evaluating the numerator of equation (6).

To evaluate the integral of equation (5), the following assumptions are made: (α) $\hbar\omega_s \ll E$ (classical plasma), (β) $|\mathbf{k}'| = |\mathbf{k}|$ (elastic scattering), (γ) the scattering is isotropic.

Furthermore, to obtain an analytical expression for the relaxation time $\tau_c(E)$, the integral in equation (5) is evaluated in accordance with the mean value theorem for integrals. Since $U_s(\mathbf{q})$ is bounded in the interval $(0, \hat{q}_s)$, it is taken at the mean value of the two limits. The maximum vector $\hat{q}_s$ is extrapolated to large wave numbers $q \approx n_{e,i}^{1/3}$ for each of the $e-e$ and $e-i$ interactions.

Accordingly, equation (6) becomes

$$\tau_c(E) = \frac{(5\sqrt{2}/3)m^{1/2}E^{3/2}}{8\pi ne^4[R_e(\hat{\varepsilon}_e) + R_i(\hat{\varepsilon}_i)]} \tag{7}$$

where

$$R_s(\hat{\varepsilon}_s) = Z_s\left(1 - \frac{2}{5}\hat{\varepsilon}_s + 4\sum_{\nu=1}^{\infty}\frac{B_{2\nu}\hat{\varepsilon}_s^{2\nu}}{(2\nu+4)(2\nu)!} + \cdots\right), \qquad s = e, i \tag{8}$$

$B_{2\nu}$ are the Bernoulli numbers, $\hat{\varepsilon}_s = \hbar\omega_s(\hat{q}_s)/k_B T$, $Z_e = -1$, and $Z_i = Z$. Here e and m are the electron charge and mass, respectively.

3. TRANSPORT COEFFICIENTS OF A MAGNETIZED PLASMA

3.1. The Field Term

The LHS of equation (2) for a magnetized plasma can be written as

$$\left.\frac{\partial f}{\partial t}\right|_{\text{field}} = \mathbf{v}\cdot\frac{\partial f_0}{\partial E}\left[\left(-\frac{E-\mu}{T} - \frac{\partial\mu}{\partial T}\right)\nabla T - e\mathbf{E}\right] + \frac{e}{m}\frac{(\mathbf{v}\wedge\mathbf{B})}{c}\cdot\frac{\partial f_1}{\partial\mathbf{v}} = -\Phi\frac{\partial f_0/\partial E}{\tau_c} \tag{9}$$

where μ is the chemical potential, ∇T the temperature gradient, and $\mathbf{B}$ the magnetic field.

The function Φ can formally be expressed as

$$\Phi = \frac{\tau_c\mathbf{v}}{1+\tau_c^2\Omega_c^2}\cdot[\mathbf{A} + \tau_c(\mathbf{\Omega}_c\wedge\mathbf{A}) + \tau_c^2\mathbf{\Omega}_c(\mathbf{\Omega}_c\cdot\mathbf{A})] \tag{10}$$

with

$$\mathbf{A} = \left(-\frac{E-\mu}{T} - \frac{\partial\mu}{\partial T}\right)\cdot\nabla T - e\mathbf{E}$$

and $\mathbf{\Omega}_c = e\mathbf{B}/mc$ is the electron gyrofrequency.

Writing $\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_\parallel$ and $\nabla T = \nabla T_\perp + \nabla T_\parallel$, the perpendicular and the parallel components of the electric and temperature fields, respectively, and assuming a cubic symmetry in the plasma, we obtain an expression for the perturbation of the distribution function f_1 [equation (4)] from equation (10) in (9):

$$f_1 = -\tau_c\varphi(-e\mathbf{E}\,\partial f_0/\partial E + \nabla T\,\partial f_0/\partial T)\cdot\mathbf{v} + \tau_c^2\varphi(-e\mathbf{E}_\perp\,\partial f_0/\partial E + \nabla T_\perp\,\partial f_0/\partial T)\cdot(\mathbf{v}\wedge\mathbf{\Omega}_c) - \tau_c^3\varphi[-e(\mathbf{E}_\parallel\cdot\mathbf{\Omega}_c)\,\partial f_0/\partial E + (\mathbf{\Omega}_c\cdot\nabla T_\parallel)\,\partial f_0/\partial T](\mathbf{\Omega}_c\cdot\mathbf{v}) \quad (11)$$

where $\varphi = 1/(1+\tau_c^2\Omega_c^2)$.

In the longitudinal direction all fields and currents are parallel and $\mathbf{\Omega}_c \wedge \mathbf{A} = 0$ and hence the transport coefficients of parallel components will be independent of the magnetic field, so that equation (10) is simply $\Phi = \tau_c \mathbf{v}\cdot\mathbf{A}$. The magnetic field has no effect on the spherical energy surfaces.

For the transverse direction we have $\Omega_c\cdot\mathbf{A} = 0$, and equation (11) becomes

$$f_1 = -\tau_c\varphi(-e\mathbf{E}_\perp\,\partial f_0/\partial E + \nabla T_\perp\,\partial f_0/\partial T)\cdot\mathbf{v} + \tau_c^2\varphi(-eE_\perp\,\partial f_0/\partial E + \nabla T_\perp\,\partial f_0/\partial T)\cdot(\mathbf{v}\wedge\mathbf{\Omega}_c) \quad (12)$$

3.2. Transport Coefficients

In the Boltzmann theory the electrical current and energy fluxes are given, respectively, by

$$\mathbf{J} = -e\int 2\frac{d^3p}{h^3}\mathbf{v}f_1(\mathbf{v}) \quad (13)$$

$$\mathbf{Q}_E = \int 2\frac{d^3p}{h^3}\frac{mv^2}{2}\mathbf{v}f_1(\mathbf{v}) \quad (14)$$

The heat current will then be given by (Spitzer and Harm, 1953; Braginskii, 1958)

$$\mathbf{Q} = \mathbf{Q}_E + (\mu/T - \partial\mu/\partial T)T\mathbf{j}/e \quad (15)$$

By substituting f_1 from (12) into (13)–(15) we obtain

$$\mathbf{E}_\perp = \mathbf{j}/\sigma_\perp + S_\perp\,\nabla T + R_\perp\mathbf{B}\wedge\mathbf{j} + N_\perp\mathbf{B}\wedge\nabla T \quad (16)$$

$$\mathbf{Q}_\perp = TS_\perp\mathbf{j}_\perp - \lambda_\perp\,\nabla_\perp T + N_\perp T\mathbf{B}\wedge\mathbf{j} + L_\perp\mathbf{B}\wedge\nabla T \quad (17)$$

This represents a complete set of transport coefficients in arbitrary magnetic fields.

The transport coefficients are now defined from equations (16) and (17) in terms of integrals involving the electron relaxation time established from the collective approach.

These are as follows.

Electrical conductivity:

$$\sigma_\perp = e^2K_{01}[1 + \Omega_c^2(K_{02}/K_{01})^2] \quad (18)$$

Thermoelectric power:

$$S_\perp = (\mu/T - \partial\mu/\partial T)/e - (eK_{11}/T\sigma_\perp) \times (1 + \Omega_c^2 K_{02}K_{12}/K_{11}K_{01}) \quad (19)$$

Hall effect:

$$R_\perp = -(e/mc)(K_{02}/K_{01}\sigma_\perp) \quad (20)$$

Nernst coefficient:

$$N_\perp = -(e^2/mc)(K_{12}/T\sigma_\perp)(1 - K_{02}K_{11}/K_{01}K_{12}) \quad (21)$$

Ettinghausen coefficient:

$$N_\perp T = -(e^2/mc)(K_{12}/\sigma_\perp)(1 - K_{02}K_{11}/K_{01}K_{12}) \quad (22)$$

Thermal conductivity:

$$\lambda_\perp = (K_{21}/T) \times (1 + \Omega_c^2 K_{12}^2/K_{21}K_{01}) - (e^2K_{11}^2/T\sigma_\perp) \times (1 + \Omega_c^4 K_{02}^2 K_{12}^2/K_{11}^2 K_{01}^2) \quad (23)$$

Leduc–Righi coefficient:

$$L_\perp = (-e/mcT)[K_{22} - (e^2K_{12}K_{11}/\sigma_\perp)(1 - K_{02}K_{11}/K_{12}K_{01}) (1 + \Omega_c^2 K_{02}K_{12}/K_{11}K_{01}) - K_{12}K_{11}/K_{01}] \quad (24)$$

Here

$$K_{ij} = \int 2\frac{d^2p}{h^3}\tau_c^j \varphi\left(\frac{v^2}{3}\right)\left(\frac{-\partial f_0}{\partial E}\right)E^i \quad (25)$$

For classical plasmas (nondegenerate electrons), $f_0(E)$ is taken to be the Maxwell distribution function in the evaluation of the coefficients K_{ij}.

3.3. Weak Magnetic Field $(\Omega_c\bar{\tau}_c)^2 \ll 1$

The coefficients defined above become

$$\sigma_\perp \simeq 8ne^2A_\tau(k_BT)^{3/2}/(\pi^{1/2}m) = 2ne^2\bar{\tau}_c/m \quad (26)$$

$$S_\perp \simeq -5k_B/2e \quad (27)$$

$$R_\perp \simeq -315\pi/512nec \quad (28)$$

$$N_\perp \simeq -2205\pi^{1/2}A_\tau k_B(k_BT)^{3/2}/64mc = -2205\pi k_B\bar{\tau}_c/256mc \quad (29)$$

$$N_\perp T \simeq -7.315\pi^{1/2}A_\tau(k_BT)^{5/2}/64mc = -7.315\pi(k_BT)\bar{\tau}_c/256mc \quad (30)$$

$$\lambda_\perp \simeq 32nA_\tau k_B(k_BT)^{5/2}/(\pi^{1/2}m) = 8nk_B(k_BT)\bar{\tau}_c/m \quad (31)$$

$$\lambda_\perp \simeq \frac{45675neA_\tau^2 k_B (k_B T)^4}{32m^2c} = \frac{45675\pi_{ne} k_B (k_B T)\bar{\tau}_e^2}{512m^2c} \tag{32}$$

where

$$A_\tau = (5\sqrt{2}/3)m^{1/2}/\{8\pi ne^4[R_e(\hat{\varepsilon}_e + R_i(\hat{\varepsilon}_i)]\} \tag{33}$$

The transport coefficients (26)–(31) are independent of the magnetic field.

3.4. Strong Magnetic Field $(\Omega_c \bar{\tau}_c)^2 \gg 1$

In this new limit the coefficients (18)–(24) become

$$\sigma_\perp \simeq 3\pi^{1/2} ne^2 A_\tau (k_B T)^{3/2}/4m = 3\pi ne^2 \bar{\tau}_c/16m \tag{34}$$

$$S_\perp \simeq -k_B/e \tag{35}$$

$$R_\perp \simeq -1/nec \tag{36}$$

$$N_\perp \simeq -2k_B/\pi^{1/2} mc\Omega_c^2 A_\tau (k_B T)^{3/2} = -8k_B \bar{\tau}_c/\pi mc\Omega_c^2 \bar{\tau}_c^2 \tag{37}$$

$$N_\perp T \simeq 2/\pi^{1/2} mc\Omega_c^2 A_\tau (k_B T)^{1/2} = -8k_B T\bar{\tau}_c/\pi mc\Omega_c^2 \bar{\tau}_c^2 \tag{38}$$

$$\begin{aligned} \lambda_\perp &\simeq 8nk_B (k_B T)^{-1/2}/(3\pi^{1/2} m\Omega_c^2 A_\tau) \\ &= (nk_B^2 T/m)\bar{\tau}_c (32/3\pi\Omega_c^2 \bar{\tau}_c^2) \end{aligned} \tag{39}$$

$$L_\perp \simeq -5nek_B^2 T/2m^2 c\Omega_c^2 = (-5nek_B^2 T/2m^2 c)(\bar{\tau}_c^2/\Omega_c^2 \bar{\tau}_c^2) \tag{40}$$

The mean relaxation time $\bar{\tau}_c$ is obtained through

$$\bar{\tau}_c = \frac{\int 2(d^3p/h^3)\tau_c(E) f_0(E)}{\int 2(d^3p/h^3) f_0(E)} = \frac{4}{\sqrt{\pi}} A_\tau (k_B T)^{3/2} \tag{41}$$

where $\tau_c(E)$ is given by equation (7).

For arbitrary magnetic field strength the coefficients K_{ij} [equation (25)] can be evaluated numerically.

4. CONCLUDING REMARKS

The main contribution of the present work is a demonstration that collective modes of a relatively dense plasma can be used as an interacting system where the electron (as a wave) can emit and absorb longitudinal plasma waves (similar to the interaction of electrons with phonons in liquid metals).

Table I. Ratio of Electrical Conductivities from the Present Model (pr) and from Lee and More (LM)[a]

T (°K)	$\sigma_{\perp pr}/\sigma_{\perp LM}$ for given values of n (cm^{-3})				
	10^{18}	10^{20}	10^{22}	10^{24}	10^{26}
10^5	1.123	0.725	0.358	Deg	Deg
10^6	1.697	1.281	0.926	0.615	Deg
10^7	2.137	1.703	1.302	0.993	0.899

[a]Good agreement is found for a wide range of temperatures and densities. Deg, degeneracy of the electrons.

Table II. Ratio of Thermal Conductivities from the Present Quantum Model (pr) and from Lee and More (LM) from a Classical Kinetic Approach[a]

T (°K)	$\lambda_{\perp pr}/\lambda_{\perp LM}$ for given values of n (cm^{-3})				
	10^{18}	10^{20}	10^{22}	10^{24}	10^{26}
10^5	2.415	1.559	0.769	Deg	Deg
10^6	3.650	2.754	1.992	1.324	Deg
10^7	4.596	3.663	2.800	2.136	1.934

[a]Deg, degeneracy of the electrons.

The results obtained are comparable in both magnitude and behavior to those obtained in classical kinetic calculations (Spitzer and Harm, 1953). The electrical conductivity [equation (26)] recovers the $(k_B T)^{3/2}$ behavior as widely observed in the experiments.

For the magnetized plasma and for a strong magnetic field strength, Tables I and II represent a comparison of the present electrical $\sigma_{\perp pr}$ [equation (34)] and thermal $\lambda_{\perp pr}$ [equation (39)] conductivities with those obtained by (Lee and More, 1984), $\sigma_{\perp LM}$ and $\lambda_{\perp LM}$, respectively.

The ratios $\sigma_{\perp pr}/\sigma_{\perp LM}$ and $\lambda_{\perp pr}/\lambda_{\perp LM}$ are of the same order of magnitude over a wide range of temperatures and densities.

Moreover, the ratio $\lambda_{\perp}/\sigma_{\perp} T$ [equations (26) and (31)] (the Wiedemann–Franz law) is recovered for a Lorentz gas, since this ratio represents precisely the ideal Lorentz number $4(k_B/e)^2$.

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